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Bound and scattering wave functions for a velocity-dependent Kisslinger potential for $l > 0$

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Abstract. Using formal scattering theory, the scattering wave functions are extrapolated to negative energies corresponding to bound-state poles. It is shown that the ratio of the normalized scattering and the corresponding bound-state wave functions, at a bound-state pole, is uniquely determined by the bound-state binding energy. This simple relation is proved analytically for an arbitrary angular momentum quantum number $l > 0$, in the presence of a velocity-dependent Kisslinger potential. The extrapolation relation is tested analytically by solving the Schrödinger equation in the p-wave case exactly for the scattering and the corresponding bound-state wave functions when the Kisslinger potential has the form of a square well. A numerical resolution of the Schrödinger equation in the p-wave case and of a square-well Kisslinger potential is carried out to investigate the range of validity of the extrapolated connection. It is found that the derived relation is satisfied best at low energies and short distances.

PACS. 03.65.Nk Scattering theory – 24.90.+d Other topics in nuclear reactions: general (restricted to new topics in section 24)

1 Introduction

It is well known that the scattering-state wave functions and the corresponding bound-state ones are proportional when the scattering wave functions are extrapolated to the positions of bound-state poles. Joachain showed that the relative normalization, the ratio of the normalized bound and scattering wave functions at a bound-state pole, depends on the form of the potential through the corresponding Jost functions and their derivatives [1]. However, a later work [2] derived a simple relation showing that the relative normalization is uniquely determined by the bound-state binding energy in the case of a local potential only. The case of a non-local but separable Yamaguchi potential was also studied and the relative normalization was found to depend on the binding energy when the binding energy is small [3]. In [4], we proved analytically that the relation derived in [2] is still valid in the presence of a velocity-dependent Kisslinger potential in the s-wave case. Here, we shall extend this relation to the velocity-dependent Kisslinger potential for $l > 0$. To test the validity of the extrapolation relation, the Schrödinger equation is solved exactly for the scattering and bound-state wave functions in the p-wave case and the Kisslinger potential is taken to be a square well. A numerical resolution of the Schrödinger equation is also carried out to investigate the range of validity of the derived relation.

The simplified relation would be useful in reactions where final-state interaction effects are important. For instance, one is able to express the cross-section for the reaction $pp \to pn\pi^+$ in terms of those for $pp \to d\pi^+$ and $pp \to pp\pi^0$ at low binding energies [5,6].

The Kisslinger potential is obtained from the equations describing the multiple scattering of particles off complex nuclei. It is an improved optical model in the sense that its form and magnitude reflect the scattering of the incident particle by the individual nucleons bound in the nucleus [7]. Using such a potential it was possible to predict the predominantly p-wave nature of the elementary pion-nucleon coherent scattering. Kisslinger theory resulted in a term proportional to

$$\nabla \cdot (\rho \nabla \psi) = \rho \nabla^2 \psi + \nabla \rho \cdot \nabla \psi. \quad (1)$$

The first term on the right is proportional to the kinetic energy, and combines with the kinetic energy term in the Schrödinger equation. However, the second term is proportional to the rate of change of density and hence it is sensitive to the diffuse edge of nuclei. This property made possible the prediction of the backward scattering of mesons by light nuclei, where the effect of the diffuse edge is important.

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2 Schrödinger equation for a velocity-dependent Kisslinger potential

Kisslinger developed a velocity-dependent potential, which may be expressed as

$$2mV(r) = U(r) + \nabla \cdot (\rho(r)\nabla), \tag{2}$$

where the reduced potential $U(r)$ and the Kisslinger term $\rho(r)$ are real, spherically symmetric functions of the radial variable r. For a particle of mass m, the Schrödinger equation for arbitrary angular momentum may be expressed as

$$v_l''(k,r) - \frac{\rho'}{1-\rho}v_l'(k,r) + \frac{1}{1-\rho} \times \left[k^2 - U(r) + \frac{\rho'}{r} - (1-\rho)\frac{l(l+1)}{r^2}\right] v_l(k,r) = 0, \tag{3}$$

where the energy of the particle is defined as $E = k^2/2m$, and the prime designates a derivative with respect to r. In the last equation the dependence of ρ on the radial variable r has been omitted for clarity. Obviously, the Kisslinger term $\rho(r)$ must be bounded away from 1. So for all r, either $\rho(r) > 1$ or $\rho(r) < 1$. In what follows we shall derive the conditions that $\rho(r)$ and $U(r)$ must satisfy so that eq. (3) has well-behaved, physically acceptable solutions.

2.1 Small r behavior

Close to the origin one may assume

$$U(r) \approx c_0 r^q, \qquad \rho(r) \approx b_0 r^p. \tag{4}$$

If $p > 0$, then $\rho(r) < 1$ for all r and eq. (3) is regular at the origin if $q \geq -2$. Using the expansion

$$v_l(k,r) = \sum_{n=0}^{\infty} a_n r^{n+s}, \tag{5}$$

and (4), eq. (3) reads

$$\sum_{n=0}^{\infty} a_n[(n+s)(n+s-1) - l(l+1)]r^{n+s-2} - pb_0 \sum_{n=0}^{\infty} a_n s r^{n+s+p-2} + k^2 \sum_{n=0}^{\infty} a_n r^{n+s} - c_0 \sum_{n=0}^{\infty} a_n r^{n+s+q} = 0. \tag{6}$$

For $q > -2$ the indicial equation reads

$$s(s-1) - l(l+1) = 0. \tag{7}$$

Consequently, in the vicinity of the origin $v_l(k,r) \to r^{l+1}$ implying that $v(k,0) = 0$, and hence it is regular at the origin. However, if $q = -2$, then by considering the coefficient of r^{s-2} it can be easily shown that physical solutions can be obtained provided

$$c_0 > -\frac{1}{4} - l(l+1). \tag{8}$$

For the case $p < 0$ then $\rho(r)$ is singular at the origin. If the velocity-dependent part is repulsive near the origin, then $\rho(r) > 1$ for all r. However, if $\rho(r)$ is attractive close to $r = 0$, then it is less than 1 for all r. In either case, the term $1 - \rho(r) \approx -b_0 r^p$ in the vicinity of the origin, and (3) is regular at the origin, provided $q - p \geq -2$. We then obtain

$$\sum_{n=0}^{\infty} a_n \left[(n+s-1)(n+s+p) - l(l+1)\right] r^{n+s-2} - \frac{k^2}{b_0}\sum_{n=0}^{\infty} a_n r^{n+s-p} + \frac{c_0}{b_0}\sum_{n=0}^{\infty} a_n r^{n+s+q-p} = 0. \tag{9}$$

Using the above equation it is possible to see that for $q - p = -2$, eq. (3) has acceptable physical solutions if

$$\frac{c_0}{b_0} \leq l(l+1) + \frac{1}{4}(1+p)^2. \tag{10}$$

However, to have wider applicability, we shall assume that $q - p > -2$. In this case the indicial equation of (9) is

$$(s-1)(s+p) - l(l+1) = 0, \tag{11}$$

and, by the Frobenious method, we have at least one solution such that $v(k,0) = 0$.

2.2 Behavior at large distances

In order to investigate the behavior of $U(r)$ and $\rho(r)$ at large distances, we write $v_l(k,r) = g_l(r)e^{ikr}$ and assume $g(r)$ to be a slowly varying function of the radial variable r. If we substitute for $v_l(k,r)$ in (3) and ignore the small term $g_l''(r)$ we get

$$ln[g_l(r)] = \int_b^{\infty} \frac{U - \rho k^2 + \rho'(ik - 1/r) + l(l+1)(1-\rho)/r^2}{2ik(1-\rho) - \rho'}, \tag{12}$$

where $b > 0$. For $g_l(r)$, and consequently $v_l(k,r)$, to be finite we require

$$\lim_{r\to\infty} U(r) = \frac{M}{r^{1+\epsilon}}, \tag{13}$$

and

$$\lim_{r\to\infty} \rho(r) = \frac{N}{r^{1+\epsilon}}, \tag{14}$$

where $\epsilon > 0$, and M, N are finite constants. This implies that at large distances both the local and the velocity-dependent parts of the potential must fall off faster than $1/r$.

3 The scattering wave function as a linear combination of Jost solutions and functions

The Kisslinger potential may be transformed into a local, but energy-dependent one by carrying out the transformation on the wave function [8]

$$v_l(k,r) = \frac{\chi_l(k,r)}{\sqrt{1-\rho(r)}}. \tag{15}$$

Substituting for $v_l(k,r)$ in eq. (3) results in

$$\chi_l''(k,r) + \left[k^2 - \frac{l(l+1)}{r^2} - U_{\rm e}(k,r)\right]\chi_l = 0, \qquad (16)$$

where $U_{\rm e}(k,r)$ is the effective local, but energy-dependent potential defined as

$$U_{\rm e}(k,r) = -\frac{1}{(1-\rho(r))} \times \left[k^2 - U(r) + \frac{\rho'(r)}{r} + \frac{\rho'^2(r)}{4(1-\rho(r))} + \frac{\rho''(r)}{2}\right]. \qquad (17)$$

Apart from the fact that the effective potential depends on the wave number k, the above is the Schrödinger equation for an arbitrary angular momentum quantum number. Following closely a similar procedure to that adopted in [4], it is possible to express $\chi_l(k,r)$ in terms of a linear combination of Jost solutions and functions, *viz*

$$\chi_l(k,r) = -\frac{1}{2ik}\frac{1}{|f_l(k)|} \times \left[f_l(-k)f_l(k,r) - (-1)^l f_l(k) f_l(-k,r)\right], \qquad (18)$$

where the Jost solutions are defined asymptotically as

$$\lim_{r\to\infty} e^{-i\pi l/2} e^{\pm ikr} f_l(\pm k, r) = 1, \qquad (19)$$

and the corresponding Jost functions are defined as

$$f_l(\pm k) = f_l(\pm k, 0) = \frac{2l+1}{(2l+1)!!}\lim_{r\to 0}(kr)^l f_l(\pm k, r). \qquad (20)$$

For $\rho(r) = 0$, the analytic properties of the Jost solutions and functions of (16) are presented in [9]. However, in our case $\rho(r)$ is not zero. The analytic properties of the Jost solutions in the complex k-plane are identical to those derived in [4], for the case $l = 0$ in the presence of a Kisslinger potential, except at $k = 0$. Provided the potential terms satisfy

$$\int_0^\infty r|U(r)|{\rm d}r < \infty, \qquad \int_0^\infty r^2|U(r)|{\rm d}r < \infty, \qquad (21)$$

and

$$\int_0^\infty r|\rho''(r)|{\rm d}r < \infty, \qquad \int_0^\infty r^2|\rho(r)|{\rm d}r < \infty, \qquad (22)$$

then $f_l(k,r)$ is analytic in the lower-half k-plane except at $k = 0$, where it has a pole of order l. The conditions in (21) are the same as those imposed on $U(r)$ in the absence of $\rho(r)$, namely that $U(r)$ diverges slower than $1/r$ at small distances and falls off faster than $1/r^3$ at infinity. However, the first condition imposed on the Kisslinger part of the potential implies that $\rho'(r)$ diverges less than $1/r$ for small r. This can be satisfied if, in the vicinity of the origin, $\rho(r) \approx b_0 r^p$, where $p > 0$. The second condition demands that $\rho(r)$ decreases faster than $1/r^3$ at large distances. The analyticity of the Jost solution $f_l(k,r)$ is extended if we impose the more stringent inequalities

$$\int_0^\infty e^{mr}|U(r)|{\rm d}r < \infty, \qquad \int_0^\infty e^{mr}|\rho(r)|{\rm d}r < \infty, \qquad (23)$$

where m is real and positive. In such a case, $f_l(k,r)$ is analytic for ${\rm Im}(k) < m/2$ except at $k = 0$. From the boundary condition (19) and the form of eq. (3) one may conclude that in the region of analyticity, including the real axis, the Jost solutions and functions satisfy the conditions

$$f_l^*(-k^*, r) = (-1)^l f_l(k,r), \qquad (24)$$

and

$$f_l^*(-k^*) = f_l(k). \qquad (25)$$

The scattering function may now be expressed as

$$v_l(k,r) = -\frac{1}{2ik}\frac{1}{\sqrt{1-\rho(r)}} \times \left[e^{-i\delta_l(k)} f_l(k,r) - (-1)^l e^{i\delta_l(k)} f_l(-k,r)\right], \qquad (26)$$

where $\delta_l(k)$ is the scattering phase shift, in terms of which the scattering matrix is defined as

$$S_l(k) = e^{2i\delta_l(k)} = \frac{f_l(k)}{f_l(-k)}. \qquad (27)$$

The symmetry properties of the Jost solutions and functions stated in (24) and (25) imply that $v_l(k,r)$ is real for real values of k. Using (19) and the fact that both the local and the velocity-dependent parts of the potential vanish faster than $1/r$ at infinity, then $v_l(k,r)$ behaves asymptotically as

$$v_l(k,r) = \frac{2l+1}{k}\sin(kr + \delta_l(k)). \qquad (28)$$

Provided the conditions in (23) apply, the scattering wave function, $v_l(k,r)$, may be analytically continued into the upper-half of the complex k-plane to the zeros of $f_l(-k)$ situated on the positive part of the imaginary axis. Such zeros are simple, and are poles of the scattering matrix $S_l(k)$ corresponding to the positions of bound states. In fact, for $k = i\lambda$ with $\lambda > 0$, we have

$$v_l(i\lambda, r) = -\frac{1}{2\lambda}\frac{1}{\sqrt{1-\rho(r)}}(-1)^l e^{i\delta_l(i\lambda)} f_l(-i\lambda, r), \qquad (29)$$

which at infinity behaves like

$$v_l(i\lambda, r) = -\frac{1}{2\lambda}(-1)^l e^{i\delta_l(i\lambda)} f_l(-i\lambda, r) \propto e^{-\lambda r}. \qquad (30)$$

The function $v_l(i\lambda, r)$ is a square integrable function as it vanishes at the origin, and decays exponentially at large r. Hence it represents a bound-state wave function.

For bound states, the Schrödinger equation in (3) transforms to

$$u_l''(r) - \frac{\rho'}{1-\rho}u_l'(r) + \frac{1}{1-\rho} \times \left[-\lambda^2 - U(r) + \frac{\rho'}{r} - (1-\rho)\frac{l(l+1)}{r^2}\right]u_l(r) = 0, \qquad (31)$$

where $u_l(r)$ is the bound-state wave function satisfying the boundary conditions

$$u_l(0) = 0, \qquad \lim_{r\to\infty} u_l(r) = (-i)^l f_l(-i\lambda, r) \propto e^{-\lambda r}. \quad (32)$$

In the vicinity of a bound-state pole, the S-matrix may be parameterized as

$$S_l(k) = e^{2i\delta_l(k)} = e^{i\pi l}\frac{[N_l G_l(k)]^2}{\lambda + ik}, \quad (33)$$

with N_l^2 being the residue at the bound-state pole. In the unitarized scattering length approximation $N_l^2 = 2\lambda$ and

$$G_l^2(k) = \frac{\lambda - ik}{2\lambda}, \quad (34)$$

and hence, at a bound-state pole, $G_l(i\lambda) = 1$. The boundary conditions imposed on the scattering and bound-state functions, together with the fact that $U(r)$ and $\rho(r)$ are both real, ensure that, for k real, both $v_l(k,r)$ and $u_l(r)$ remain real for all r. It is then possible to analytically continue $v_l(k,r)$ in k to the position of a bound-state pole $k = i\lambda$ on the positive imaginary axis. However, special attention must be paid to the singularity structure of $e^{i\delta_l(k)}$ as it has a branch cut at the pole. Using eq. (33) at the position of a bound-state pole, we have

$$\lim_{k\to i\lambda}[\sqrt{2\lambda(\lambda^2 + k^2)}v_l(k,r)] = -[\sqrt{\lambda + ik}e^{i\delta_l(k)}]_{k=i\lambda} f_l(-i\lambda, r) = -N_l u_l(r). \quad (35)$$

In what follows we shall show that N_l is uniquely determined by the normalization of the bound-state wave function.

If we multiply (3) by $u_l(r)$ and (31) by $v_l(k,r)$ and then rearrange, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}r}\{(1 - \rho(r))[u_l'(r)v_l(k,r) - u_l(r)v_l'(k,r)]\} = (\lambda^2 + k^2)u_l(r)v_l(k,r). \quad (36)$$

Integrating the above and using the fact that $u_l(r)$ and $v_l(k,r)$ vanish at the origin leads to

$$(1 - \rho(r))[u_l'(r)v_l(k,r) - u_l(r)v_l'(k,r)] = \int_0^r (\lambda^2 + k^2)u_l(r')v_l(k,r')\mathrm{d}r'. \quad (37)$$

In the limit $r \to \infty$ both sides of the above equation vanish. In order to a void this, we define

$$w_l(k,r) = 2ik\sqrt{\lambda + ik}\; v_l(k,r), \quad (38)$$

which by (33) has the limit at the pole

$$w_l(i\lambda, r) = N_l u_l(r). \quad (39)$$

Multiplying (37) by $2ik\sqrt{\lambda + ik}$ and differentiating the resulting with respect to k readily gives

$$(1 - \rho(r))\left[u_l'(r)\dot{w}_l(k,r) - u_l(l)\dot{w}_l'(k,r)\right] = \int_0^r \left[(\lambda^2 + k^2)\dot{w}_l(k,r') + 2kw_l(k,r')\right]u_l(r')\mathrm{d}r', \quad (40)$$

where the dot refers to differentiation with respect to k. By taking the limit $k \to i\lambda$ followed by $r \to \infty$, the first term in the integrand vanishes and the right-hand side simplifies to

$$2i\lambda N_l \int_0^r u_l^2(r')\mathrm{d}r'. \quad (41)$$

Making use of (26), (33) and (38) in the neighborhood of a bound-state pole gives

$$\sqrt{1 - \rho(r)}w_l(k,r) = N_l G_l(k)(-i)^l f_l(-k,r) - (-i)^l\frac{(\lambda + ik)}{N_l G_l(k)} f_l(k,r), \quad (42)$$

which when differentiated with respect to k gives

$$\sqrt{1 - \rho(r)}\dot{w}(k,r) = (-i)^l N_l \dot{G}_l(k) f_l(-k,r) + (-i)^l N_l G_l(k)\dot{f}_l(-k,r) - (-1)^l\frac{i^{l+1}}{N_l G_l(k)} f_l(k,r) - (-i)^l\frac{\lambda + ik}{N_l G_l(k)}\dot{f}_l(k,r) + (-i)^l(\lambda + ik)\frac{\dot{G}_l(k)}{N_l G_l^2(k)} f_l(k,r). \quad (43)$$

At the position of a bound-state pole, we have

$$\sqrt{1 - \rho(r)}w_l(i\lambda, r) = (-i)^l N_l f_l(-i\lambda, r), \quad (44)$$

and

$$\sqrt{1 - \rho(r)}\dot{w}_l(i\lambda, r) = (-i)^l N_l \dot{G}_l(k) f_l(-i\lambda, r) + (-i)^l N_l \dot{f}_l(-i\lambda, r) - (-1)^l\frac{i^{l+1}}{N_l} f_l(i\lambda, r). \quad (45)$$

The Jost solutions $f_l(\pm i\lambda, r) \propto e^{\pm\lambda r}$ at infinity, then the only surviving term on the righ-hand side of the last equation is that proportional to $f_l(i\lambda, r)$. Equating both sides of (40) and noting that $\rho(r)$ vanishes at infinity gives

$$1 = N_l^2\int_0^\infty u_l^2(r)\mathrm{d}r. \quad (46)$$

The above proves that N_l is actually the normalization constant of the bound-state wave function. The last result may be rewritten in the form

$$\lim_{k\to i\lambda}\left[\sqrt{2\lambda(\lambda^2 + k^2)}\left(\frac{\lambda}{k}\right)^l v_l(k,r)\right] = -(-i)^l u_{l,\lambda}(r), \quad (47)$$

where $u_{l,\lambda}(r)$ is the normalized bound-state wave function. The factor λ/k has been introduced to take out the explicit threshold behavior and leads to and extra overall phase factor as stated in the last result.

4 Analytical test of the extrapolation relation

For simplicity, we shall consider the case $l = 1$ and solve the Schrödinger equation when the Kisslinger potential has the form of a spherically symmetric square well. In such a case, exact solutions may be obtained that may provide a valuable clue as to the validity of the derived relation in the presence of a Kisslinger potential. The assumed forms of the potentials are

$$U(r) = -U_0\theta(a - r), \tag{48}$$

$$\rho(r) = A\theta(a - r), \tag{49}$$

where a is the common radius of both potentials. The boundary conditions are such that the wave functions are continuous at $r = a$, but due to the effect of $\rho(r)$ at the sharp boundary, the condition on the derivatives becomes

$$(1 - A)\psi'(r < a) = \psi'(r > a). \tag{50}$$

The solutions for the bound-state wave functions for $l = 1$ are

$$R_{\mathrm{i}}(r) = Nj_1(\kappa r), \qquad r \leq a\,, \tag{51}$$

$$R_{\mathrm{e}}(r) = -N\sin(\kappa a)e^{\lambda a}h_1(i\lambda r), \qquad r \geq a\,, \tag{52}$$

where $j_1(x)$ and $h_1(x)$ are the Bessel and Hankel functions respectively for $l = 1$. Further

$$\kappa = \sqrt{\frac{U_0 - \lambda^2}{1 - A}}, \tag{53}$$

and $R_{\mathrm{i}}(r)$, $R_{\mathrm{e}}(r)$ are the internal and external radial wave functions, respectively. The normalization constant $N = \sqrt{2N_1/a^3N_2}$, where

$$\begin{aligned} N_1 = {} & (4 + a^4\kappa^4)(1 + \lambda a)^2A^2 + a^4(\kappa^2 + \lambda^2) \\ & \times(\lambda^2 + \kappa^2(1 + a\lambda)^2) + 2a^2A\{2\lambda^2(1 + \lambda a) \\ & -a^2\kappa^4(1+\lambda a)^2+\kappa^2(2+4\lambda a+3a^2\lambda^2+a^3\lambda^3)\}, \end{aligned} \tag{54}$$

$$\begin{aligned} N_2 = {} & a^2(\kappa^2 + \lambda^2)(3 + 3\lambda a + \lambda^2a^2) \\ & +A^2\{-2 - 4\lambda a + 3a^3\kappa^2\lambda \\ & +a^4\kappa^2\lambda^2 + a^2(3\kappa^2 - 2\lambda^2)] + A\{6 + 12\lambda a - 2a^4\kappa^2\lambda^2 \\ & +a^2(-6\kappa^2 + 7\lambda^2) + a^3(-6\kappa^2\lambda + \lambda^3)\}, \end{aligned} \tag{55}$$

and the transcendental equation that determines the bound-state energies is

$$\frac{\cot(\kappa a)}{\kappa a} - \frac{1}{\kappa^2a^2} = \frac{(1 - A)(1 + \lambda a)}{\lambda^2a^2 + 2A(1 + \lambda a)}. \tag{56}$$

Finally, the scattering wave functions are

$$R_{\mathrm{i}}(K, r) = 3Cj_1(Kr), \qquad r \leq a\,, \tag{57}$$

$$\begin{aligned} R_{\mathrm{e}}(k, r) = {} & 3\{\cos\delta_1(k)j_1(kr) \\ & -\sin\delta_1(k)n_1(kr)\}, \qquad r \geq a\,, \end{aligned} \tag{58}$$

where

$$K = \sqrt{\frac{U_0 + k^2}{1 - A}}. \tag{59}$$

and

$$\begin{aligned} C^{-2} = {} & \left\{\left(1-A+\frac{2A}{K^2a^2}\right)\sin Ka-\frac{2A}{Ka}\cos Ka\right\}^2+k^2a^2 \\ & \times\left\{\left(\frac{2A}{k^2a^2}-1\right)\frac{\cos Ka}{Ka}\right. \\ & \left.+\left(\frac{1}{K^2a^2}-\frac{1}{k^2a^2}+\frac{A}{k^2a^2}-\frac{2A}{a^4k^2K^2}\right)\sin Ka\right\}^2. \end{aligned} \tag{60}$$

An expression for $\tan\delta_l$ is obtained by matching the internal and external scattering wave functions at the boundary according to (50).

Substituting the scattering wave functions above on the left-hand side of (47), and then using l' Hôpital's rule, one recovers the right-hand side, thus analytically proving the correctness of the relation.

5 Numerical resolution of the Schrödinger equation

In this section we shall carry out a numerical resolution of the Schrödinger equation to determine the range of validity of the extrapolation relation. The bound-state energies (and hence the values of λ) are determined using (56). For $A = 0.5$, and $U_0 = 6.5$, only one bound state is sustained corresponding to $\lambda = 0.164$. The first allowed bound state occurs at a higher potential, $U_0 = 6.5$, than that needed in the s-wave case. This is reasonable from a physical point of view. The $l = 1$ term is interpreted as an additional potential energy, which corresponds to a repulsive "centrifugal force". This suggests that a particle possessing angular momentum requires a stronger attractive potential to bind it than a particle with zero angular momentum. Figure 1 shows the bound-state wave function for $\lambda = 0.164$ represented as a solid line and the scattering wave function modified according to

$$v_1(k, r) \approx -\frac{k}{\lambda\sqrt{2\lambda(\lambda^2 + k^2)}}u_{l,\lambda}(r), \tag{61}$$

The dashed line corresponds to $k = 0.17$, while the dash-dotted and dotted lines correspond to $k = 0.3$ and 0.4, respectively. Clearly, in the vicinity of a bound-state pole the agreement between the bound-state wave function and the modified scattering one is best at short distances. The cross-over point, which was also observed in [4], serves to limit the deviation from the extrapolation relation. The discontinuity in the derivatives of the wave functions at the sharp boundary, due to the Kisslinger term, is clear.

When the local potential is reduced to -21.62, two bound states are possible. The first corresponds to $\lambda = 0.164$, while for the other $\lambda = 3.66$. In fig. 2, the solid curve is the bound-state wave function corresponding to $U_0 = 21.62$ and $\lambda = 0.164$. The dashed line represents the modified scattering wave function. In the vicinity of the bound-state pole, the agreement is better here than that observed in fig. 1. This is so, since the value of $U_0 = 21.62$

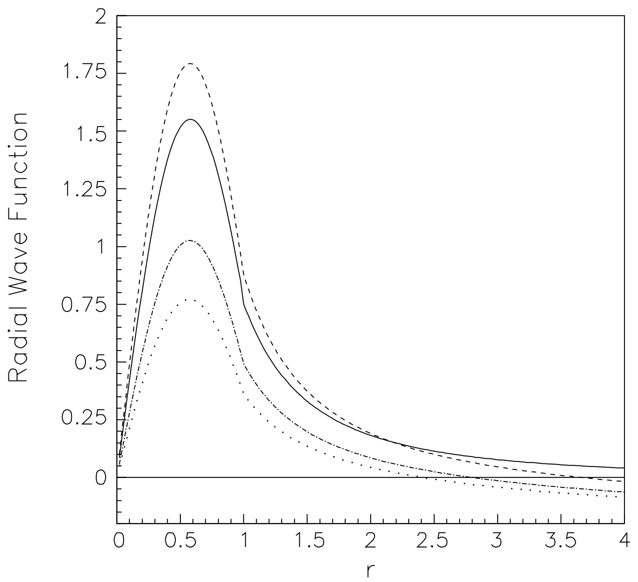

Fig. 1. The bound-state wave function (solid line) corresponding to $U_0 = 6.5$, $\lambda = 0.164$ and a Kisslinger term $A = 0.5$. The modified scattering wave function according to (61) is plotted corresponding to $k = 0.17$ (dashed line), $k = 0.3$ (dash-dotted line) and $k = 0.4$ (dotted line).

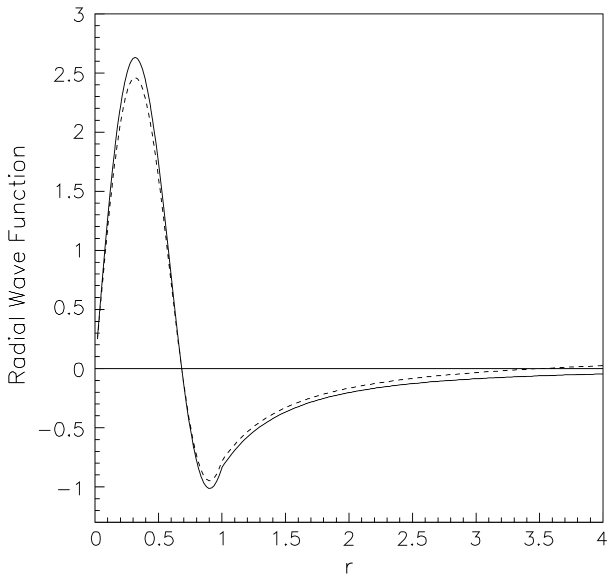

Fig. 2. The bound-state wave function (solid line) plotted for $U_0 = 21.62$, $\lambda = 0.164$ and $A = 0.5$. The corresponding modified scattering wave function is plotted for $k = 0.17$ (dashed line).

is large compared to $\lambda = 0.164$ and $k = 0.17$. Consequently, the values of κ and K are very close and hence better agreement. For the higher-energy bound state $\lambda = 3.66$, the deviation from the derived relation is quite large as the values of κ and K differ markedly.

In principle, one may derive an analytical expression for the deviation from the extrapolation relation as was done in [4]. But the presence of the velocity-dependent term in the case $l = 1$ complicates the algebra and one may lose physical insight with such expressions. However, by examining figs. 1 and 2, it can be concluded that such corrections are small at small distances and low energies.

6 Discussion and conclusion

In this work, we have shown analytically that the scattering and corresponding bound-state wave functions are linked through a simple relation, which is independent of the details of the potential, at the position of a bound-state pole. This has been done for $l > 0$ in the presence of a velocity-dependent Kisslinger potential. The extrapolation relation was tested analytically by solving exactly the Schrödinger equation for $l = 1$, when the Kisslinger potential took the form of a spherically symmetric square well. Using such solutions on the left-hand side of (47), and with the help of l' Hôpital's rule, one readily recovers the right-hand side, thus showing the correctness of the extrapolation relation at the bound-state pole. Further, a numerical resolution of the Schrödinger equation was carried out and, as can be seen in figs. 1 and 2, the agreement between the bound-state wave functions and the corresponding extrapolated scattering ones is better at small distances and low energies.

In the s-wave case, the binding energies are lower than the corresponding ones in the p-wave case. Therefore, the agreement between the bound and modified scattering-state wave functions in the s-wave case is better than that for the p-wave in the vicinity of a bound-state pole.

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